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Portfolio Tax Trading with Carry-Over Losses*

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Abstract

We study portfolio choice with multiple stocks and capital gain taxation assuming that capital losses can only offset current or future realized capital gains. We show, through backtesting using empirical distributions, that optimal equity holdings over an extended period are significantly lower on average than benchmark holdings suggested in the literature. Using Value and Growth or Small and Large portfolios, the backtests show that allocations remain persistently under-diversified. Carry-over losses have large economic significance since they can dramatically shrink the no-trade region. Finally, the backtested economic cost of incorrectly modeling capital losses is at least 8 percent of lifetime wealth.

Keywords: portfolio choice, capital gain taxation, limited use of capital losses, carry-over losses

JEL Classification: G11, H20

1 Introduction

According to the tax code, realized capital losses on equity can only be used to offset current or future realized capital gains. However, to simplify the dimensionality of the problem, studies of portfolio choice with capital gain taxation typically assume that the use of capital losses is unrestricted. The goal of this paper is to understand how assumptions on the use of capital losses drive portfolio choice with capital gain taxation. Using the time-series and empirical distribution of the S&P 500 Index, we show in a series of backtests that optimal equity holdings with *limited use of losses* (LUL) remain significantly lower than optimal equity holdings with *full use of losses* (FUL) over an extended period of time. Averaging these backtests shows that the LUL and FUL holdings equate only after 30 years. Unused capital (or carry-over) losses play a key role in delaying convergence, since they significantly alter the trading strategy and the no-trade region under the LUL setting.

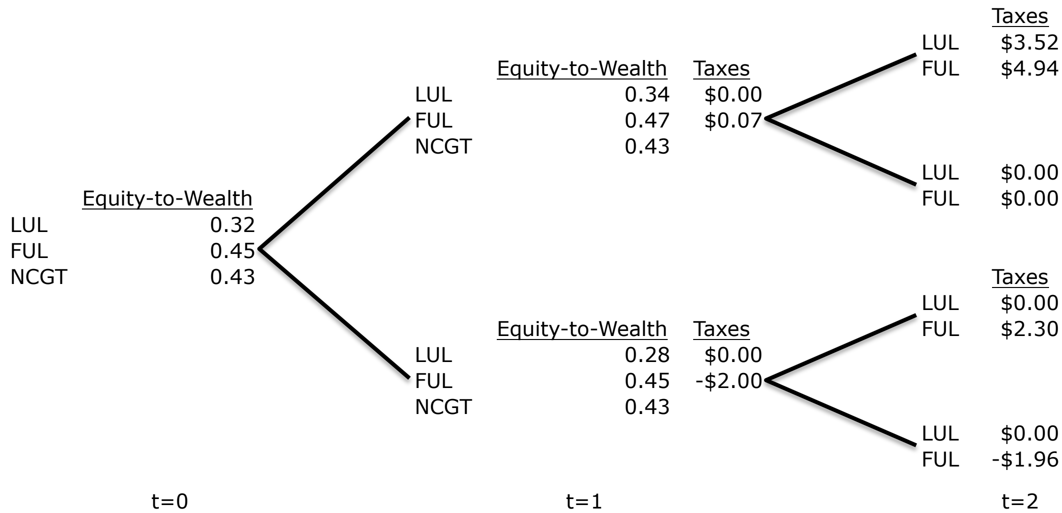
Due to the dimensionality and singularity in the dynamics of the portfolio choice problem with capital gain taxes, the academic literature commonly assumes that the use of capital losses is unrestricted. If capital losses are larger than capital gains in a period, the investor receives a tax rebate that cushions the downside of holding equity. This simplification is helpful as then an unused carry-over loss in the portfolio can never occur; hence, it does not need to be tracked. While it is convenient to reduce the number of state variables, the simplification comes at a cost. Namely, tax rebates boost the demand for equity relative to the LUL case. What is surprising is the magnitude of the difference: in some cases the benefit of a tax rebate is so large that an FUL investor holds even more equity than an untaxed investor (NCGT) when the portfolio contains no capital gain. Even when both the FUL and the LUL investors have embedded capital gains and elevated (locked-in) equity holdings, i.e., the tax-induced trading costs exceeds the cost from holding equity above the untaxed benchmark, expected tax rebates can elevate the equity-to-wealth ratio of the FUL investor above the already elevated equity-to-wealth ratio of the LUL investor. Only when the tax lock-in is so severe that expected rebates and expected carry-over losses are insignificant, the equity-to-wealth ratios of the FUL and LUL investors equate and move in lockstep.

We illustrate the impact of alternative assumptions on the tax treatment of capital losses in a simple two-period portfolio choice problem with a stock and a bond. Up (down) moves in a binomial tree denote stock price increases (decreases), where there are two trading dates and a final date where the portfolio is consumed. The investor maximizes the constant relative risk aversion utility of after-tax final period wealth. Initially, the investor has an endowment of $100 with no embedded capital gains or losses.[1] Figure 1 summarizes the optimal portfolio choice expressed as an equity-to-wealth ratio and capital gain taxes paid through the binomial tree under the LUL- and FUL-based capital gain tax systems and when no capital gain taxes are paid (NCGT).

Figure 1 shows that at $t = 0$ the LUL investor chooses an equity-to-wealth ratio of 0.32, which is significantly below the constant equity-to-wealth ratio of 0.43 for the NCGT investor. Also, the FUL investor's initial equity-to-wealth ratio, at 0.45, is higher than that of the NCGT investor. From an after-tax risk-return tradeoff perspective, an allocation above the NCGT allocation is possible. If the tax reduces the volatility of after-tax returns

[1] Additional details are provided in the Internet Appendix A.

Figure 1: **Motivating Example**



more than the after-tax risk premium, the after-tax Sharpe ratio is pushed higher implying a higher demand for equity than the NCGT investor. However, from inspecting the FUL case, this intuition is misleading. The FUL investor's increased equity demand is driven by the prospect of artificially cushioning the impact of a stock price drop through a tax rebate. Specifically, if the stock price drops at $t = 1$, a tax rebate of $2 is collected which immediately increases the FUL investor's wealth. If the stock price increases at $t = 1$, both the LUL and the FUL investors are overexposed to equity with embedded capital gain. Given the LUL investor started with a smaller investment, the equity-to-wealth ratio of 0.34 is again smaller than for the NCGT investor. The FUL investor still holds the most equity with an equity-to-wealth ratio of 0.47. This simple short-horizon example shows that it can be quite important how capital losses are treated in optimal portfolio choice problems with capital gain taxation.

We use the test region iterative contraction method, described in Yang and Tompaidis (2013), to solve a lifetime dynamic portfolio choice problem with capital gain taxation. This approach allows us to handle several endogenous state variables and singularities that are due to capital gain taxation and the limited use of losses.[2] Therefore, our analysis differs

[2]While alternative numerical solution approaches exist, such as the one in Brandt, Goyal, Santa-Clara, and Stroud (2005), Gallmeyer, Kaniel, and Tompaidis (2006), and Garlappi and Skoulakis (2008), we were

from the literature in the way we model the use of capital losses.[3] In particular, we seek to understand how LUL strategies evolve over time in realistic settings through a series of backtests using the S&P 500 Index and popular investment strategies such as Small and Large or Value and Growth stock portfolios.

What is wrong with the intuition that after a few years the equity holdings of the FUL and LUL investors are locked-in and the treatment of the use of capital losses becomes irrelevant? Nothing, except it is not pervasive. In several backtests, after only six years both the LUL and FUL are locked-in, the LUL investor does not have any carry-over loss, and expected tax rebates do not play a significant role in the portfolio of the FUL investor. In these cases, the FUL and LUL equity-to-wealth ratios equate after six years and from then on move in lockstep until the end of the backtest. Only a Japanese-style stock market meltdown can decouple these equity-to-wealth ratios. However, the data tell a more balanced story than the above intuition suggests as we also see examples where it takes significantly longer before the FUL and LUL investors' equity-to-wealth ratios equate. In our backtests, it takes on average 30 years before the equity-to-wealth ratios equate. Furthermore, the wedge between the FUL and LUL investors' equity-to-wealth ratios is not only long-lived, but can also be large. We present examples where the LUL equity-to-wealth ratio remains below the FUL equity-to-wealth ratio by at least 10 percent for at least 5 years and by at least 5 percent for more than 20 years. These results are conservative for a variety of reasons. The current U.S. capital gain tax rate used in our analysis is low relative to historical U.S capital gain tax rates and relative to capital gain tax rates in many other countries. Risk aversion can be lower than what we use. Capital gain taxes are not forgiven at death in Canada or Europe. Plus, we use a short horizon for the bequest motive. Otherwise, the wedge between LUL and FUL equity-to-wealth ratios would be even larger and longer lived.

A general pattern in the paths of equity-to-wealth ratios exists across different backtest

unable to implement them efficiently for the two-stock case with the limited use of losses.

[3]See the single stock model with capital gain taxes of Dammon, Spatt, and Zhang (2001b) and the multiple stock model with capital gain taxes of Gallmeyer, Kaniel, and Tompaidis (2006), both of which permit the full or unrestricted use of capital losses.

periods. Upon entering the stock market, the LUL investor holds less equity than the NCGT investor, while the FUL investor holds more equity than the NCGT investor as tax rebates from capital losses reduce downside risk. On average, the stock market appreciates over time and the FUL and LUL investors' equity-to-wealth ratios end up higher than their own optimal equity holdings without embedded gains or losses. At the same time, the wedge between the FUL and LUL equity-to-wealth ratio becomes smaller as both investors are facing more capital gains. This is because the FUL and LUL investors only differ in the way that a capital loss is treated and, therefore, the larger are embedded gains in a portfolio the less likely is that a loss can occur. Hence, FUL and LUL portfolios become more and more similar.

Analyzing the conditional optimal trading strategies characterized by the no-trade regions sheds further light on the differences and similarities between the FUL and LUL investors. The capital gain tax friction generates a no-trade region in the optimal trading strategy, where the investor only sells (buys) equity when the equity holding is higher (lower) than a upper (lower) bound and optimally chooses not to rebalance in between the bounds. When the basis-to-price ratio is close to one, the investors face small embedded gains. For this case, both the lower and upper bounds of the FUL investor are above the upper bound of the LUL investor. Only when the basis-to-price ratio is sufficiently low, which can happen once the investors become locked-in in their positions due to significant embedded capital gains, do the two no-trade regions overlap. From inspecting the optimal no-trade regions, a 1 percent (10 percent) carry-over loss-to-wealth ratio shrinks the no-trade region by one (two) third(s) relative to the case when there is no loss in the portfolio. Therefore, our findings show that carry-over losses have a large, and economically significant, impact on asset allocations under capital gain taxation.

Through a thought experiment, we measure the cost for a hypothetical investor (FUL(LUL)), who follows the FUL investment strategy, but faces the LUL treatment of the U.S. government. The FUL(LUL) investor sells more equity and thus pays more capital gain taxes and his portfolio contains larger and more persistent carry-over losses than the LUL investor. All of this is driven by the elevated equity-to-wealth ratio of the FUL investor, which the FUL(LUL) investor replicates from the literature. During stock market booms, the FUL(LUL) investor's consumption and wealth increases in lockstep with the FUL investor's consumption and wealth. During periods when the stock market declines, the consumption and wealth of the FUL(LUL) investor can fall more than those of the LUL investor. Hence, the FUL(LUL) investor's consumption and wealth can exhibit higher volatility than those of the FUL, LUL, and NCGT investors. It is not surprising that over time the expected cumulative utility of the FUL(LUL) investor declines gradually and significantly below that of the LUL investor. When young, the FUL(LUL) investor enjoys a roughly 4 percent higher wealth equivalent than the NCGT investor, meaning he prefers to be taxed just as the FUL investor. This is driven by the fact that the FUL investor consumes more than the NCGT investor while expecting to receive some tax rebates over his lifetime. The effect of mistakenly consuming more, holding elevated equity position, and hoping to receive tax rebates accumulates over time. When old, the wealth equivalent of the FUL(LUL) investor declines to roughly 8 percent below that of the LUL investor and 10 percent below the NCGT investor. This cost rises to 20 percent of the investor's entire lifetime wealth at current European tax rates.

In two-stock backtests with Small and Large or Value and Growth stocks, we see large deviations in the portfolio of the LUL investor relative to the unconstrained optima with no embedded capital gains and losses. For example, simultaneously the equity-to-wealth ratio for one stock increases by as much as 10 percent above and the equity-to-wealth ratio for the other stock decreases by roughly 7 percent below the unconstrained equity-to-wealth ratios. Such large and persistent deviations from the unconstrained equity-to-wealth ratios that have no embedded capital gains and losses are driven by the fact that both stocks are locked-in, but one has more embedded gain than the other. Thus, it is not tax optimal to sell the more locked-in stock. The purchase of the other stock is prevented through an already elevated total equity-to-wealth ratio. In addition, tax optimal Small and Large or Value and Growth portfolios reconfirm that the optimal total equity holdings with the limited use of losses over extended periods of time remain significantly lower than the optimal total equity holdings with the full use of losses.

It should not go unnoticed that the optimization problems studied are at the computational forefront of portfolio choice theory. For each stock, we track two state variables: the stock holding and the weighted average purchase price. In addition, tracking the unused carry-over loss under the LUL assumption requires one more state variable. Thus, the one-stock FUL case has two state variables, the one-stock LUL case has three state variables, the two-stock FUL case has four state variables, and the two-stock LUL case has five state variables excluding time. All state variables are endogenous. As a run-time benchmark based on our computing resources, the two-stock LUL portfolio choice problem takes approximately 90 hours to solve using 100 CPUs (2.66GHz) running in parallel on supercomputers.[4] We stress that the method used to solve the portfolio choice problem can be applied to other asset allocation problems.

Our results do not depend on how we model stock market returns or if we explicitly model that individual investors can offset up to $3,000 of taxable income per year with realized capital losses. We solve for optimal portfolios under three stock return assumptions: a normal distribution, an empirical distribution using all available data, and an empirical distribution constructed from realized data only. Even with a normal distribution, we find that LUL optimal equity holdings remain significantly lower than FUL optimal equity holdings. Large negative returns, present in the empirical distributions, widen the gap between FUL and LUL optimal equity holdings. We focus on the empirical distribution using all available data as this leads to no-trade regions that depend only on age but not on time. Restricting the empirical distribution to only realized data results in even greater and more persistent differences between FUL and LUL investors in the backtests. In a final extension, we solve

[4]Modeling a third stock requires two more state variables and amplifies the computational cost by at least 10 times using a quasi-random grid, which is parsimonious as compared to a regular grid.

for optimal portfolios incorporating the tax provision that individual investors can offset up to $\$3,000$ of taxable income per year with realized capital losses (LUL3K). As it turns out, the $\$3,000$ tax provision matters mostly for young investors with low income and wealth. Yet, even for 20-year-old investors who start with no wealth and an income of $\$50,000$ per year,[5] the average initial LUL3K optimal equity holding is 63 percent compared to 56 percent for the LUL investor and 73 percent for the FUL investor. By the age of 30, the difference between the average LUL3K and LUL holdings narrows down to 1.5 percent, but the FUL holdings are still 5.5 percent above the LUL3K holdings.

While our modeling of capital gain taxation through limited use of losses is more realistic when compared to the full use of losses, modeling individuals' investment behavior remains a challenge. According to the Department of the Treasury (Office of Tax Analysis), realized capital gains between 1954 and 2009 amount to 1.76 to 7.35 percent of GDP, are roughly 3.5 times larger than corporate capital gains (1955-1999), and roughly yield 3 times more tax revenue than dividends (2000-2005). Matching these empirical facts requires modeling additional incentives to sell equity, such as inflows through saved returns on capital and inheritance; outflows such as purchase of real estate; college tuition for offspring and dis-saving over retirement; anticipating changes in the capital gain tax rate over time; and perhaps behavioral biases. Each additional assumption complicates the formulation of the problem and makes it difficult to solve with current optimization tools.

2 The Consumption-Portfolio Problem

The investor chooses an optimal consumption and investment policy at trading dates $t = 0, ..., T$. Our assumptions concerning the exogenous price system, taxation, and the investor's

[5]The median pre-tax household income is $\$46,700$. Within the middle quintile of the 2013 income distribution, only 9.2 percent of families directly own stocks and only 6.1 percent of families own pooled investment funds (which could be stocks, bonds, etc.). Therefore, only 15.3 percent of households in this quintile of the income distribution potentially hold stocks. See Federal Reserve Bulletin, "Changes in U.S. Family Finances from 2010 to 2013: Evidence from the Survey of Consumer Finances," vol. 100, no. 4 (Sept. 2014).

portfolio problem are outlined below.

2.1 Security Market

The set of financial assets available to the investor consists of a riskless money market and multiple stocks. The money market pays a continuously-compounded pre-tax rate of return, while stocks pay dividends.

2.2 Taxation

Dividends and interest income are taxed as ordinary income on the date they are paid at rates τ_D and τ_I, respectively. Realized capital gains and losses are subject to a constant capital gain tax rate τ_C.

The tax basis used for computing realized capital gains or losses is calculated as a weighted-average purchase price.[6] When an investor dies, capital gain taxes are forgiven and tax bases of stocks owned reset to current market prices. This is consistent with the reset provision in the U.S. tax code. Dividend and interest taxes are still paid at the time of death. While investors are allowed to "wash sell" and immediately rebalance after they realize capital losses, they are precluded from shorting the stock which eliminates a "shorting against the box" transaction to avoid paying capital gain taxes.[7]

[6]The U.S. tax code allows for a choice between weighted-average price and exact identification of the shares to be sold. The Canadian and European tax codes use the weighted-average price rule. While choosing to sell the shares with the smallest embedded gains using the exact identification rule is beneficial to the investor, solving for the optimal investment strategy becomes numerically intractable for a large number of trading periods given the dimension of the state variable increases with time (Dybvig and Koo, 1996; Hur, 2001; DeMiguel and Uppal, 2005). However, for parameterizations similar to those in this paper, DeMiguel and Uppal (2005) numerically show that the certainty-equivalent wealth loss using the weighted-average price basis rule as compared to the exact identification rule is small. In our setting, using the exact basis rule would most likely further amplify the role of carry-over losses for the LUL strategy and tax rebates for the FUL strategy as the capital losses will be realized sooner.

[7]A wash-sale is a sale of a financial security with an embedded capital loss and a proximate repurchase (within 30 days before or after the sale) of the same or substantially similar security. We permit wash-sales as highly correlated substitute securities, that are not considered substantially similar, typically exist in most stock markets allowing an investor to re-establish a position with a similar risk-return profile after a capital loss. For an analysis of possible portfolio effects of wash-sales when adequate substitute securities do not exist, see Jensen and Marekwica (2011). A shorting against the box transaction involves short selling securities that the investor owns to defer tax on capital gains. The Taxpayer Relief Act of 1997 no longer

A common assumption regarding the capital gain tax code in the portfolio choice literature is that there are no restrictions on the use of capital losses. It has the computational advantage that capital losses are never carried over and hence the investor does not need to keep track of an additional state variable. Tax codes, however, restrict the use of capital losses. We define the two cases as follows.

Definition 1. *Under the full use of capital losses (FUL) case, an investor faces no restrictions on the use of realized capital losses. When realized capital losses are larger than realized capital gains in a period, the remaining capital losses generate a tax rebate that can be immediately invested.*[8]

Definition 2. *Under the limited use of capital losses (LUL) case, an investor can only use realized capital losses to offset current realized capital gains. Unused capital losses can be carried forward indefinitely to future trading dates.*

We assume that FUL and LUL investors immediately realize all capital losses each period even if they are not used.[9] Additionally, our analysis does not distinguish between differential taxation of long and short-term capital gains since our investors trade annually.[10]

2.3 Investor's Portfolio Choice Optimization Problem

To finance consumption, the investor trades in risky stocks and the money market. The setting we have in mind is one where a taxable investor trades individual stocks or exchange traded funds (ETFs).[11] Given an initial equity endowment, a consumption and security

allows delaying taxation through shorting.

[8]Definition 1 is used in several papers that study portfolio choice with capital gain taxes (Constantinides (1983); Dammon, Spatt, and Zhang (2001a,b, 2004); Garlappi, Naik, and Slive (2001); Hur (2001); DeMiguel and Uppal (2005); Gallmeyer, Kaniel, and Tompaidis (2006)).

[9]The no-arbitrage analysis in Gallmeyer and Srivastava (2011) shows that, under the LUL case, an investor is indifferent between realizing an unused capital loss or carrying it forward.

[10]Dammon and Spatt (1996) explores optimal tax-timing in the presence of long and short-term capital gain tax rates. Dai, Liu, Yang, and Zhong (2015) reexamines optimal tax-timing in the presence of asymmetric long-term and short-term capital gain tax rates with limited deductibility of capital losses where they approximate the average holding time of equity for tractability.

[11]To isolate the effect of the LUL assumption, we abstract away from investing in mutual funds where unrealized capital gain concerns can also be important. Like mutual funds, ETFs must pass unrealized

trading policy is *admissible* if it is self-financing, involves no short selling of stocks, and leads to nonnegative wealth over the investor's lifetime. The investor lives at most T periods and faces a positive probability of death each period.

The investor's objective is to maximize the expected utility of real lifetime consumption and a time-of-death bequest motive by choosing an admissible consumption-trading strategy given an initial endowment. The utility function for consumption and terminal wealth is of the constant relative risk aversion form with a relative risk aversion coefficient γ.

Using the principle of dynamic programming, the Bellman equation for the investor's optimization problem, derived in Internet Appendix B, can be solved numerically by backward induction starting at time T. The numerical algorithm is described in Internet Appendix C.

3 Dynamic Tax Trading Strategies

3.1 Parameterizations

The investor begins trading at age 20 and can live to a maximum of 100 years.[12] He has a time discount parameter $\beta = 0.96$ and his relative risk aversion coefficient is $\gamma = 5$. The bequest motive is set such that the investor plans to provide an equal amount of payment each year in real terms for 20 years to his heirs using his liquidated wealth at death.

The tax rates used are set to roughly match those faced by a wealthy investor under the U.S. tax code. Specifically, interest is taxed at the investor's marginal income tax rate $\tau_I = 35\%$; dividends are taxed at $\tau_D = 15\%$; the capital gain tax rate is set to the long-term rate $\tau_C = 20\%$;[13] and capital gain taxes are forgiven at the investor's death.

capital gains onto investors generated by portfolio rebalancing. However, many ETFs substantially reduce and in some cases eliminate unrealized capital gains. This is achieved through a "redemption in kind" process described in Poterba and Shoven (2002).

[12]The probability that an investor lives up to period $t < T$ is given by a survival function, calibrated to the 1990 U.S. Life Table, compiled by the National Center for Health Statistics where we assume period $t = 0$ corresponds to age 20 and period $T = 80$ corresponds to age 100.

[13]The U.S. Tax Relief and Reform Act of 2003 changed several features of the tax code with respect to investments. Specifically, the long-term capital gain tax rate dropped from $\tau_C = 20\%$ to $\tau_C = 15\%$ for most individuals. Dividend taxation switched from being linked to investor's marginal income tax rate to a flat

We solve one stock optimal portfolios using the empirical distribution of the S&P 500 Index and two stock optimal portfolios using the empirical distributions of Small and Large or Value and Growth stock portfolios, based on the sorts from Kenneth French's website. As compared with normally-distributed returns, the empirical distribution captures the negative skewness in returns, which implies a higher probability of large negative returns and lower probability of large positive returns.[14]

3.2 Benchmarks

To understand the role of the LUL assumption on portfolio choice, we compare it to three benchmark portfolio choice problems. The first benchmark is the case when the investor faces no capital gain taxation, abbreviated as NCGT. In this benchmark, the investor still pays dividend and interest taxes. The second benchmark is the FUL case when the investor faces no restrictions on the use of capital losses. The third benchmark is an investor who implements the optimal FUL allocation but is subject to the limited use of capital losses tax treatment, abbreviated as FUL(LUL). Specifically, this benchmark is constructed by assuming that the FUL(LUL) investor keeps investing and consuming at the same rate as a FUL investor even though he never receives a tax rebate but can only use capital losses to offset current or future capital gains just as the LUL investor. The FUL(LUL) benchmark helps measure the economic cost of the FUL rule, a simplified taxation rule, that an investor might follow to reduce the complexity of the portfolio optimization problem.

rate of $\tau_D = 15\%$. The 2006 Tax Reconciliation Act extended these rates to be effective until 2010. This was further extended through 2012 in legislation passed by Congress. From 2013, rates reverted to $\tau_C = 20\%$. Beginning in 2013 there is a contribution to Medicare of 3.8% for the lesser of the net investment income or the excess of the adjusted gross income above \$200,000 (\$250,000 for a joint return or half thereof if married taxpayers file separate returns). For a comprehensive summary of U.S. capital gain tax rates through time, see Figure 1 in Sialm (2009).

[14] A drawback of our constant opportunity set assumption is that it does not allow for any autocorrelation structure over time. Yet, all the historical annual return series used show very low autocorrelations which are statistically insignificant at a 5% significance level.

3.3 The Optimal Tax Trading Strategy

At each trading date, an investor's optimal trading strategy depends on the entering stock position and the entering tax basis-to-price ratio. The basis-to-price ratio is the weighted average purchase price divided by the current stock price. With a basis-to-price ratio above one, there is an embedded loss in the stock position. With a basis-to-price ratio below one, there is an embedded gain in the stock position. The lower the basis-to-price ratio, the more gains are embedded in the stock position and, thus, the larger is the tax friction.

Unconstrained Optimum and No-Trade Region. The investor's optimal trading strategy can be characterized by the unconstrained optimum and the no-trade region as shown in the top left plot of Figure 2. Capital gain taxation affects investors' portfolio choice through two channels: the immediate tax consequences from current portfolio rebalancing and the expectation about the after-tax risk-return profile of the investment opportunity set. When the basis-to-price ratio equals 1, there is no embedded gain or loss, thus the current portfolio rebalancing does not trigger a capital gains tax. Still, optimal allocations are affected by the presence of capital gains taxes through the after-tax risk-return profile of the investment opportunity set. Specifically, the LUL investor balances the return benefits from investing in the stock market with the unavoidable capital gain taxes by reducing the optimal equity share even relative to the NCGT investor at time $t = 0$ (or whenever the basis-to-price ratio is equal to or above one) when the adjustment does not lead to capital gain taxation. The investor's optimal portfolio, referred to as the "unconstrained optimum" henceforth, solely reflects his risk appetite and expected after-tax opportunity set. For allocations above the unconstrained optimum, the investor has an incentive to sell stock until reaching the unconstrained optimum. Below it, the investor has an incentive to buy stock back to the unconstrained optimum. When the basis-to-price ratio is below 1, due to the immediate tax friction, there is a no-trade region around the unconstrained optimum allocation. The area covered by the no-trade region reflects the extent of the tax friction, that is, the lower is the basis-to-price ratio the larger is the no-trade region around the unconstrained optimum allocation. Above the no-trade region, the investor sells stock until reaching the upper boundary. Below the no-trade region, the investor buys stock until reaching the lower boundary. Inside the no-trade region, the investor optimally refrains from trading.[15] When the basis-to-price ratio is above 1, there is an embedded loss. Then, the optimal strategy is to liquidate the stock position to realize the loss, and buy the stock back up to the unconstrained optimum allocation, as discussed in Subsection 2.2.

No-Trade Region: FUL vs LUL. The top right plot of Figure 2 compares the no-trade regions of the FUL investor and the LUL investor with zero carry-over loss at age 20 (and 80). The most important observations is that, at the same age, the FUL no-trade region is above the LUL no-trade region for a wide range of basis-to-price ratios. This is because under the FUL assumption the tax rebate cushions the downside of holding equity, thereby the treatment of losses boosts the FUL investor's overall demand for equity. In addition, the difference between FUL and LUL is the largest at the unconstrained optimum allocation, where the basis-to-price ratio equals 1, and diminishes as the ratio decreases from 1 to 0. The intuition is that as the basis-to-price ratio decreases, there are more embedded gains, thus a smaller chance to face a loss in the future. Since FUL and LUL investor's differ only in the way a loss is treated, the difference between FUL and LUL no-trade regions diminishes when losses become unlikely. For very low basis-to-price ratios, the boundaries of the no-trade regions of FUL and LUL converge.

No-Trade Region: Age Effect. From Figure 2's top right plot, we see that both the FUL and LUL no-trade regions at age 80 are lower than the no-trade regions at age 20, which suggests that the optimal equity holdings are decreasing in the investor's age. This result seems to contradict Dammon, Spatt, and Zhang (2001b), who show that the optimal equity holdings are increasing in investor age under a similar setup as our FUL investor. What drives the difference is the level of the bequest motive. For the case with a higher

[15] Portfolio choice with capital gain taxes can be compared to portfolio choice with transaction costs; see for example Dumas and Luciano (1991), Liu and Loewenstein (2002), the multiple risky asset analysis in Liu (2004), and the literature therein. Specifically, from this literature we know that there exists a no-trade region, in which investors optimally refrain from trading towards the unconstrained optimal portfolio.

bequest motive mentioned in Subsection 3.8 and shown in the Internet Appendix, we do observe that investor's optimal equity holdings are increasing in investor age. The intuition is that capital gain taxes are forgiven at death, so the investor has an incentive to defer realization of capital gains by holding higher equity positions while facing embedded gains. As we increase the bequest motive, this incentive becomes stronger.

No-Trade Region: Carry-Over Loss Effect. The carry-over loss amount has a large impact on the LUL no-trade region as shown in Figure 2's bottom right plot. With a carry-over loss of only 1% of the investor's wealth, the size of the no-trade region is significantly reduced as compared to the no-trade region with zero carry-over loss. Intuitively, a carry-over loss reduces the tax friction and allows the investor to trade closer to the unconstrained optimum. Increasing the carry-over loss to 10% of wealth, the size of the no-trade region is further reduced. With an infinite carry-over loss, the upper and lower boundaries of the LUL no-trade region collapse into one horizontal line, which equates with the optimal holding of the NCGT investor. With an infinite amount of carry-over losses, the LUL investor never pays capital gain taxes.

At a basis-to-price ratio of 1, the stock allocation of the FUL investor is above that of the NCGT investor and, in turn, above that of the LUL investor with a finite carry-over loss. We would expect the FUL investor to hold more equity than the LUL investor because tax rebates increase the attractiveness of a stock investment to the FUL investor. Nevertheless, it is still a surprise to see that expected tax rebates push the FUL investor's equity holding above that of the NCGT investor.

No-Trade Region: State Dependency. The difference between the FUL and LUL no-trade regions or, equivalently, the FUL and LUL optimal trading policies is state dependent. In general, the difference between FUL and LUL is larger when the basis-to-price ratio is higher and/or the carry-over loss is larger. For example, in state A in the bottom left plot of Figure 2, the FUL investor buys shares to reach the lower boundary of the FUL no-trade region. The LUL investor, instead, sells shares until the equity-to-wealth ratio reaches the upper boundary of the LUL no-trade region. After rebalancing, the FUL investor holds roughly 10 percent more wealth in stock than the LUL investor. In state B, the FUL investor does not trade while the LUL investor sells shares until the equity-to-wealth ratio reaches the upper boundary of the LUL no-trade region. In state C, neither the FUL nor the LUL investor trades. In state D, the FUL and LUL investors sell stock until their allocations reach the same overlapping boundary, which results in no difference in their stock holdings after trade.

Convergence of FUL and LUL. Over time the stock market appreciates and the investor faces more and more embedded gains. As a result, the basis-to-price ratio on average decreases from 1 towards 0 and the difference between FUL and LUL allocations decline significantly. The crucial question is how long it takes to move from the high basis-to-price region to the low basis-to-price region. To answer this question, we start with a backtest of the optimal trading strategies using the S&P 500 index.

3.4 Backtesting with the S&P 500 Index: Equity-to-Wealth Ratios

We consider a sequence of investors reaching age 20 in each of the years between 1927 and 1999. When investors turn 20, they begin investing in equity and in the riskless money market. Each investor is endowed with 100 dollars. Using the optimal portfolios computed from the empirical distribution of the S&P 500 Index,[16] we let each investor experience the realized paths of the S&P 500 Index, dividends, inflation, and interest rates.

Figure 3 plots realized paths of equity-to-wealth ratios of selected backtest windows, with the only difference being the year when a 20-year-old investor starts to invest. From the top right plot of Figure 3, we see that a LUL investor, who enters the market in 1950, optimally chooses to invest 38 percent of his wealth in the S&P 500 Index, while the FUL and NCGT

[16] The Internet Appendix presents selected tabulated examples of optimal equity-to-wealth ratios ($\overline{\pi}(t)$) conditional on the beginning period equity-to-wealth and basis-to-price ratios ($\underline{\pi}(t)$ and $b(t)$), at ages 20 and 80.

investors start with equity-to-wealth ratios of 53 percent and 42 percent, respectively. As time passes, investors dynamically adjust their equity holdings, where the NCGT investor's equity holding stays constant, the FUL investor's equity holding fluctuates between 50 and 56 percent, and the LUL investor's equity holding steadily increases towards the FUL investor's equity holding. It only takes six years before the FUL and LUL equity-to-wealth ratios equate. From this example, one might conclude that the treatment of capital losses is not crucial for how real life equity-to-wealth ratios evolve.

Yet, the top left (1929 to 1959) and the middle left (1965 to 1995) plots of Figure 3 show examples where the LUL equity-to-wealth ratio remains below the FUL equity-to-wealth ratio for periods of twenty years or more. In these examples, the LUL equity-to-wealth ratio remains 10 percent below the FUL equity-to-wealth ratio for at least five years (middle plots) and remains on average five percent below the FUL equity-to-wealth ratio for at least twenty years (top left and middle left plots). From these examples, it is more difficult to conclude that the treatment of capital losses is inconsequential for how real life equity-to-wealth ratios evolve over time.

To mitigate cherry-picking concerns, the bottom left plot displays the average path of equity-to-wealth ratios by investor age. The average is computed from 58 investors, or roughly three generations, who enter the stock market at age 20 in 1927, 1928, ..., and 1984, respectively. The 58th investor enters the equity market in 1984 and reaches age 50 in 2014. This exercise shows that there exists a general pattern in the paths of equity-to-wealth ratios across different backtest periods. At the beginning of each backtest period, the LUL investor holds less equity than the NCGT investor. The reduced equity-to-wealth ratio results from the trade off between harvesting the equity risk premium and minimizing capital gains taxation. On the contrary, the FUL investor holds more equity than the NCGT investor because the tax rebates on capital losses reduce the downside risk of equity. As time passes, the NCGT investor holds a constant fraction of his wealth in equity, the FUL investor's equity-to-wealth ratio fluctuates roughly around a constant level, and the

LUL investor's equity-to-wealth ratio appears to gradually increase to the level of the FUL investor, although the speed of convergence varies. The periods of 1950 to 1965 and 1980 to 1995 are examples of fast convergence between LUL and FUL. The period of 1965 to 1995 is an example roughly on par with the average convergence in the bottom left plot, which is 30 years. Examples of slow convergence are the first backtest window, 1929 to 1959, and the last backtest window, 1999 to 2014.

We close the discussion of how the use of capital losses drives equity-to-wealth ratios with the last backtest window of Figure 3. It shows the most recent episode with the first rebalancing taking place in January 1999 and the last one in January 2014. The wide initial gap between the FUL and LUL equity-to-wealth ratios demonstrates a modest sign of convergence of roughly 5 percent over this 15-year period.

3.5 Backtesting with the S&P 500 Index: Trading and Taxes

To better understand tax-induced trading in the backtests, in particular why the FUL and LUL equity-to-wealth ratios over some periods converge faster than over others, we depict an investor's optimal trading strategy by showing the optimal no-trade regions and how the equity-to-wealth ratios evolve around the no-trade regions in Figure 4. The figure shows two representative backtest windows from Section 3.4. First, the period 1950 to 1965 gives an example of fast convergence. Second, the period 1999 to 2014 gives an example of slow convergence.[17] For the same backtest windows, we present in Figure 5 the evolutions of the basis-to-price ratio, the carry-over loss to wealth ratio, and the cumulative capital gain taxes to demonstrate the tax consequences of trading over the realized return path.

Figure 4 shows the trajectories of the investor's equity holdings before and after rebalancing each year on top of the no-trade regions shown in Figure 2. Comparing these trajectories with the no-trade regions, we see that the upper boundaries of the no-trade regions are more relevant for investor's trading decisions in a stock market that appreciates on average. As

[17]Figure 4 only shows the first 10 years of period 1950-1965 and the first 8 years of period 1999-2007.

the basis-to-price ratio decreases, both the FUL and LUL upper boundaries increase. As both investors become more locked-in, they are more reluctant to sell their elevated equity positions due to higher capital gain taxes per unit of stock sold. In addition, the FUL upper boundary has a much smaller negative slope than the LUL upper boundary since when the basis-to-price ratio is close to one, the FUL upper boundary is above the LUL one. The difference in shape implies that on average the FUL investor's equity-to-wealth ratio is above the no-trade region immediately after he enters the stock market and remains above it most of the time. The LUL investor's equity-to-wealth ratio instead, on average, moves into the no-trade region and stays there over several trading periods after he enters the stock market. Over time, the average basis-to-price ratio declines for both investors. Eventually, the LUL investor's equity-to-wealth ratio also increases above the upper boundary of the no-trade region and from there on we see that the LUL investor sells equity shares just as the FUL investor. Once the upper boundary of the LUL and FUL no-trade regions coincide, the equity-to-wealth ratios equate and move in lockstep.

Inspecting the equity-to-wealth ratios in the left plots of Figure 4 of the FUL dynamic tax trading strategies over the backtest period from 1950 to 1960, all entering equity-to-wealth ratios are above the no-trade region, except in 1958. Hence in all but one year, the exiting equity-to-wealth ratio is lower than the entering one, implying that the FUL investor is selling equity, realizing capital gains, and paying capital gain taxes as shown in the bottom left plot of Figure 5. The exiting equity-to-wealth ratios gradually increases over time as the upper boundary of the no-trade region slightly increases for lower basis-to-price ratios. The FUL investor is locked-in from the start and gradually becomes even more locked-in. In contrast to the FUL investor, the LUL investor enters the no-trade region just after entering the stock market and remains there until 1955, which is the first year when the LUL investor sells equity. In 1956, just after 6 years, the equity-to-wealth ratios of both investors are almost equal. From 1957 on, even in the no-trade region, the equity-to-wealth ratios move in lock-step. This is entirely driven by the overlap between the upper boundary of the LUL and FUL no-trade regions, which occurs slightly below a basis-to-price ratio of 0.4. Over the entire 10 years, neither of the investors ever purchases equity as entering equity-to-wealth ratios are never below the no-trade regions.

The 1999 to 2007 backtest is starkly different as it includes a stock market crash. Over three successive years from 2000 to 2002, the stock market depreciates and the basis-to-price ratios are pushed up from below 1 with embedded gains to above 1 with embedded losses. As shown in Figure 5, both the FUL and LUL investors optimally realize their capital losses. While the FUL investor collects tax rebates evidenced by the decrease in cumulative taxes, the LUL investor accumulates carry-over losses roughly equal to 10 percent of wealth. Starting from 2003, the stock market recovers and keeps appreciating until end of 2007. Over this period, armed with an unused carry-over loss the LUL investor's upper boundary of the no-trade region is pushed down significantly as shown in Figure 4. As a result, the LUL investor starts selling equity in 2004 at a rate higher than what he would sell without a carry-over loss. Specifically, the investor keeps selling in 2005, 2006, and 2007, while he would choose not to trade if there was no carry-over loss. Thus, the non-zero carry-over loss implies a significant and persistent gap between the upper boundaries of the FUL and LUL no-trade regions. This results in a significant and persistent gap in equity-to-wealth ratios as both the FUL and LUL investors keep selling equity back to their own upper boundaries after 2004. A carry-over loss slows down the convergence between the FUL and LUL equity-to-wealth ratios.

Briefly, to complete the analysis of these backtests, Figure 5 plots investors' tax basis, the carry-over loss to wealth ratio, and the cumulative capital gain taxes. In both backtests, all taxed investors show almost identical tax bases, given the tax basis is computed as a weighted average purchase price. Only the purchase of equity can alter the tax basis, but not the sale of equity. Over the period of 1950 to 1965, the large amount of stock purchases only occur at the beginning of the backtest period. Afterwards, taxed investors keep selling stock most of the time and purchase stock only a few times in very small amounts. Over the period of 1999 to 2014, the FUL and LUL investors buy and sell stock synchronously by almost the same amount.

3.6 Backtesting with the S&P 500 Index: Wealth and Consumption

The consequences of the limited use of capital losses are not limited to investors' equity holdings and capital gain taxes, but are also reflected in the investors' wealth and consumption. We plot realized paths of investors' wealth and consumption in dollars, wealth and consumption scaled by the wealth and consumption of the NCGT benchmark, respectively, and volatilities of wealth and consumption for the period 1999 to 2014 in Figure 6. Figures showing wealth and consumption over other backtested periods are presented in the accompanying Internet Appendix.

From the top and middle left plots of Figure 6, we see that the wealth paths implied by different tax trading strategies all closely follow market cycles, where the relative differences between the investors are highlighted through scaling by the wealth of the NCGT benchmark. The wealth implied by the FUL tax trading strategy is higher than all other strategies at the end of the backtest period. It is also higher than the other strategies at market peaks and remains highest most of the time over the backtest period. This wealth difference is driven by the FUL investor's higher equity-to-wealth ratios relative to the LUL and NCGT investors, which matters when returns are positive on average, and by tax rebates when returns are negative, which are available only to the FUL investor. The FUL(LUL) investor, who always maintains an equity-to-wealth ratio identical to that implied by the FUL strategy, has the second largest wealth at the end of the backtest period. However, the wealth of the FUL(LUL) investor shows the highest volatility and over periods when the stock market declines it is lower than for all other strategies. This is because the FUL(LUL) investor holds as much equity as the FUL investor but is not cushioned by the tax rebates during market downturns.

The market cycles in the wealth paths carry over to the consumption paths, as shown in the top and middle right plots of Figure 6. Although investors smooth consumption in a countercyclical fashion relative to the wealth dynamics, considerable variations in the consumption paths remain, as indicated by the consumption volatility in the bottom-right plot.[18] The FUL(LUL) investor experiences the highest consumption volatility, while the LUL investor shows the lowest consumption volatility over the entire backtest period. While we do frequently observe in the other backtests that the consumption volatility of the NCGT strategy is lower than that of the LUL strategy, in almost all backtests we observe that the FUL(LUL) strategy shows the largest consumption volatility. Importantly, the difference between the consumption volatility of the FUL(LUL)- and the LUL-based investment provides a first, albeit indirect, glance into the real life cost of following the FUL strategy when the use of losses is restricted.

3.7 The Economic Costs of the LUL and the FUL Cases

While the dynamics of investors' wealth and consumption implied by optimal tax trading are interesting in their own right, it is important to develop a better understanding of the economic costs of the use of capital losses. We adopt a wealth equivalent measure similar to the one in Gallmeyer, Kaniel, and Tompaidis (2006). We construct $50,000$ simulation paths by randomly sampling from the joint historical annual returns of the S&P 500 Index, risk-free interest rate, and inflation rate. Each sampled path has 80 periods, corresponding to an investment experience from age 20 to 100. Starting with a zero carry-over loss and no embedded gain and loss, we simulate forward the realized utility from consumption along each path following the optimal trading strategies of a FUL investor, a LUL investor, a NCGT investor, and a FUL investor subject to the LUL tax rule (FUL(LUL)). We calculate the cumulative utility along each path by summing up the realized utility starting from age 20. The expected cumulative utility is the average over all $50,000$ paths at a particular age, which

[18] Volatility is computed by estimating a GARCH(1,1) model with a constant mean term.

is shown in the top left plot of Figure 7. For easier interpretation, we calculate the wealth equivalent of the averaged cumulative utility at each age relative to the NCGT investor at the same age, which is shown in the top right plot of Figure 7. The resulting wealth equivalent ratio is always 1 for the NCGT investor at all ages and a deviation from 1 indicates a utility loss (negative deviation) or gain (positive deviation) measured relative to the NCGT investor. At age 100, our wealth equivalent measure summarizes the investor's lifetime utility loss or gain; it is equivalent to the wealth equivalent measure in Gallmeyer, Kaniel, and Tompaidis (2006). Before age 100, the right plot in Figure 7 shows the time series of the utility loss or gain relative to the NCGT benchmark, which is crucial for the comparison between FUL and FUL(LUL), where the difference in realized utility from consumption accumulate over investor's age.

The top left plot in Figure 7 shows that the average cumulative utility of the LUL investor closely follows the one of the NCGT investor, but always lies below it due to the tax friction. Even when the LUL investor ages, there is only a slight increase in the difference between his cumulative utility and that of the NCGT investor. Due to the tax rebates, the FUL investor's average cumulative utility is larger than the one of the NCGT investor and the difference increases over time. The tax rebates increase consumption and wealth artificially, which increases average cumulative utility. In contrast, the average cumulative utility of the FUL(LUL) strategy slowly declines by age relative to the FUL strategy due to the lack of tax rebates, unused carry-over losses, and higher volatilities of wealth and consumption. The cost of ignoring the limitation on carry-over loss accumulates over time. Eventually, the FUL(LUL) benchmark shows the lowest average cumulative utility, which is more than 35 percent below that of the LUL investor at age 100.

For easier interpretation of the economic significance, we calculate the wealth equivalent from the average cumulative utility relative to the NCGT investor, shown in the top right plot of Figure 7. Over his life, the FUL investor is roughly 4 percent better off than the NCGT investor at the same age. He prefers to be taxed and would be willing to pay for it.

The LUL investor is roughly 2 percent worse off than the NCGT investor. Economically, the difference of 6 percent between FUL and LUL is quite large. The FUL(LUL) investor starts out just as the FUL investor but by age 40, his wealth equivalent ratio based on average cumulative utility coincides with the one of the NCGT investor. By age 50, it coincides with that of the LUL investor. By age 100, the wealth equivalent ratio of the FUL(LUL) investor is 8 percent below the LUL investor, 10 percent below the NCGT investor, and 14 percent below the FUL investor. In other words, ignoring the limited use of capital losses built into the tax codes all over the world will cost an individual investor on average 8 percent of his entire lifetime wealth.

Although returns are not very cyclical, the consumption volatility implied by optimal tax trading can be higher in recessions. Indeed, from the bottom plots of Figure 7, where we jointly sample from the S&P 500 Index and NBER recessions and average only across recession years, we see that welfare costs increase in recessions. In particular, the FUL(LUL) investor is by far more negatively affected by recessions than the three other investors.

3.8 Robustness

Several robustness checks of our one stock backtest are presented in the Internet Appendix. To understand the impact of the capital gain tax rate, we also study the Capital Gain Tax 30% Case. A 30% rate roughly equals the 28% rate imposed after the U.S. 1986 Tax Reform Act and is consistent with the long-term capital gain tax rate paid in many European countries. For example, the capital gain tax rates in Denmark, Finland, Norway, and Sweden are currently 27%, 30%, 28%, and 30%, respectively. In 2009, Germany's individual capital gain tax rate rose to approximately 28% from 0%.[19] To illustrate a case where stock holdings decrease for the NCGT investor and hence the dollar value of tax-loss selling decreases for the FUL investor, the Higher Risk Aversion Case assumes that γ increases to 10. The No Tax Forgiveness at Death Case assumes capital gain taxes are assessed when the investor

[19] The German capital gain tax rate is 25% plus a church tax and tax to finance the five eastern states of Germany.

dies, a feature consistent with Canadian and European tax codes. The Bequest Motive 100 Years Case shows the sensitivity of our main analysis to the horizon of the bequest motive. The Normal Distribution Case shows the sensitivity of our analysis relative to the empirical distribution. Finally, the Empirical Distribution II Case, which at any point in time samples only from realized returns, quantifies the potential bias from including all returns in the portfolio allocation decision.[20]

We find that a higher capital gains tax rate, a lower risk aversion, no tax forgiveness at death, and a longer horizon for the bequest motive all increase the initial difference between LUL and FUL portfolio strategies and slow down the convergence of the FUL and LUL strategies over time. A higher capital gain tax rate,[21] no tax forgiveness at death, and a longer horizon for the bequest motive all directly amplify the impact of the tax friction. A lower risk aversion results in more equity holdings, which leads to more embedded gains and losses and a larger impact from imposing tax frictions on those gains and losses. Relative to the normal distribution, the empirical distribution increases the initial difference between LUL and FUL portfolio strategies and slows down the convergence of the FUL and LUL strategies over time. This is because the normal distribution understates the probability of large negative returns as shown in Figure 2 of the Internet Appendix. Using the Empirical Distribution II Case, which assigns a large probability to large negative returns for quite some time after the Great Depression, we see a further increase in the initial difference between LUL and FUL portfolio strategies as well as a slowdown in the convergence of the FUL and LUL strategies over time.

All our robustness checks suggest that the main parametrization employed in Subsection 3.1 is conservative. Therefore, the analyses in Subsections 3.4, 3.7, and 3.9 provide lower bounds on the differences between LUL and FUL as well as LUL and FUL(LUL).

[20] In the Empirical Distribution II Case, an investor in year t determines his optimal trading strategy given the information set containing realized returns up to year $t-1$. While this is more realistic, it comes with the cost that the no-trade region depends not only on age but also on time.

[21] For example, after increasing the capital gain tax rate to 30%, the wealth equivalent of the FUL(LUL) investor by age 100 is 20 percent below the LUL investor, 22 percent below the NCGT investor, and 29 percent below the FUL investor.

3.9 Backtesting with Two Risky Assets

Due to the challenges of dimensionality and singularity, it is very difficult to numerically solve a long-horizon portfolio-choice problem with the limited use of capital losses for more than two stocks. Even for the two-stock case, it is still challenging to apply numerical solution approaches such as the one in Brandt, Goyal, Santa-Clara, and Stroud (2005), Gallmeyer, Kaniel, and Tompaidis (2006), and Garlappi and Skoulakis (2008). Instead, we use the test region iterative contraction method, developed in Yang and Tompaidis (2013), to solve the optimization problem.[22]

Similar to the backtests with the S&P 500 Index, we consider a series of investors who start trading at age 20. We solve optimal tax trading for well known investment strategies, such as, the portfolio of Small and Large and the portfolio of Value and Growth stocks. For Small and Large, we use data from 1945 to 2013. For Value and Growth, we use data from 1985 to 2013. These data samples imply positive weights in the strategies over the entire period.[23] Using the optimal portfolios computed from the empirical distribution of i) Small and Large and ii) Value and Growth, we present the evolution of equity-to-wealth ratios and other quantities from 1965 to 1995 and 1999 to 2014 in Figures 8 and 9 for Small and Large and from 1999 to 2014 for Value and Growth in Figure 10.

Total Equity-to-Wealth Ratios. Figures 8-10 confirm that there is a large and persistent wedge between the total equity-to-wealth ratios, shown in the top right plots, of the

[22] We note that Gallmeyer, Kaniel, and Tompaidis (2006) is based on a random search algorithm, which is reliable for the one stock LUL case and the two stock FUL case but requires relatively large computational costs. Our method solves the first order conditions and more efficiently achieves the same level of accuracy. The Internet Appendix shows that the test region iterative contraction method and the method in Gallmeyer, Kaniel, and Tompaidis (2006) produce equity-to-wealth ratios that are indistinguishable for the one stock case with the limited use of losses.

[23] One of the two "stocks" in the Small and Large portfolio and the Value and Growth portfolio exhibits a significantly lower before tax expected return than the other stock. The after tax expected returns, which matter for the equity-to-wealth ratios, are even lower. Correlations between the two stocks are high, independently of the time period length of the backtests, implying that shorting can be optimal. Since we preclude shorting, it implies that equity-to-wealth ratios can be zero. Since it is not possible to judge whether an equity-to-wealth ratio jumping back and forth between 0 and 0.02 is real or a numerical error pattern, we search for a period that ensures that the weights on both stocks in the backtests stay away from the zero bound at all times over the entire backtest period.

LUL and FUL investors. In our backtested strategies, convergence takes 15 years on average. While it is not possible to directly compare the total equity-to-wealth ratios from any two-stock example with any one-stock example, the two-asset parameters in the Internet Appendix show that Small, Large, Value, and Growth have higher realized returns than the S&P 500 Index in the data samples we use. Therefore, the basis-to-price ratios of Small, Large, Value, and Growth decline faster than for the S&P 500 Index, which speeds up convergence. This also implies that we are again conservative in our analysis as it is difficult to find investment strategies that are more successful than Small, Large, Value, and Growth.

Capital Gain Taxes in Portfolios with Several Stocks. If we can ascertain that current capital gains and losses cancel out in a portfolio with a number of individual stocks, then the optimal equity-to-wealth ratio of the LUL investor will be close to his unconstrained optimum, which is lower than the NCGT benchmark, and in turn, lower than the FUL benchmark. The longer one can maintain a portfolio in which capital gains and losses cancel out or, at least, cancel out to a large extent, the longer one can keep the total equity-to-wealth ratio close to the unconstrained optimum, and the more persistent is the wedge between FUL and LUL total equity-to-wealth ratios.

Trade-off between Diversification and Optimal Tax Trading. Next we inspect the time-series properties of the respective weights in Small and Large and Value and Growth shown in the top left and top middle plots of Figures 8 - 10. First, in the time-series, we see large deviations of the LUL investor's equity-to-wealth ratios from the initial LUL equity-to-wealth ratios at age 20 and from the equity-to-wealth ratios used by the untaxed investor. Such deviations can be as large as 10 percent of wealth. Frequently, the deviations go in opposite directions for the two "stocks." For example from Figure 8, we see that between 1980 and 1990, the LUL investor's equity-to-wealth ratio in Small stocks is more than 10 percent higher relative to the equity-to-wealth ratio chosen by the NCGT investor and the equity-to-wealth ratio in Large stocks is roughly 7 percent below the equity-to-wealth ratio chosen by the NCGT investor. Second, the FUL and LUL equity-to-wealth ratios in each stock can have any ordering, i.e., LUL above FUL or FUL above LUL, even though at the portfolio level the total LUL equity-to-wealth ratio is below or the same as the total FUL equity-to-wealth ratio in the one-stock case. Third, while the equity-to-wealth ratios of FUL and LUL investors converge at the portfolio level, they do not necessarily converge at the stock level, at least not over the same period of time.

The above observations relate to the fact that there is a trade off between total equity exposure and portfolio diversification. The desire to control total equity risk through the total equity-to-wealth ratio is at least as significant as the desire to optimally diversify. Specifically, from the basis-to-price ratios (middle left and middle middle plots) and stock shares (bottom left and bottom middle plots) in Figure 8, it is evident that the LUL investor sells Small and Large shares every year starting in 1975. From 1980 on, there is no carry-over loss in the portfolio and thus each sale involves paying capital gain taxes. After 20 years, the cumulative capital gain tax reaches 18 dollars. Right after 1975, the Small stock shows higher tax trading costs as it has a lower basis-to-price ratio (middle left plot) because the realized returns of Small stocks are on average higher than those of Large stocks. For the same reason, the equity-to-wealth ratio in the Small stock deviates more and more from the optimal equity-to-wealth ratio without capital gains and losses. To prevent further deterioration of the diversification in the portfolio, the LUL investor sells more Small stock than Large stock, which is evident from the steeper drop in stock shares for the Small stock in the bottom left plot.[24] Although the Large stock shows a significantly lower equity-to-wealth ratio over this period than the optimal equity-to-wealth ratio without capital gains and losses, we still see that shares are sold to not let the total equity-to-wealth allocation increase by too much.

The FUL investor's equity-to-wealth ratios demonstrate a similar trade-off between total equity exposure and portfolio diversification in all three backtests. Additionally, the FUL investor holds elevated positions relative to the LUL investor in one or, sometimes, both

[24]The slope of the stock shares is informative here as for both stock shares the range from low to high equals 0.3.

stocks and always in total equity. For example, from the top middle plot of Figure 8 we see that the equity-to-wealth ratio for Large stocks is 6 to 7 percent above the equity-to-wealth ratio of the LUL and NCGT investors. Interestingly, the bottom middle plot shows that after 1977 the FUL investor's stock shares in Large stocks is basically flat. Only after 1990, we see that shares in Large stocks decline to reduce the over-allocation to Large stocks.

Carry-Over Loss Effect. As for the one-stock case using the S&P 500 Index, we also consider the backtest period from 1999 to 2014, Figures 9 and 10. Large stocks, likely by definition, but certainly over the years from 1999 to 2014, show return dynamics that are substantially similar to the S&P 500 Index. This is confirmed by the basis-to-price ratio of the Large stock that follows a pattern comparable to the evolution for the S&P 500 Index basis-to-price ratio over the same backtest period. As a result, the Large stock also preserves a similar trading pattern as the S&P 500, shown in the bottom middle plot of Figure 9. What differentiates the backtest with the S&P 500 Index is that the average allocation to Large stock is smaller than to the S&P 500. After going through the same market crash, a smaller tax rebate for the FUL investor and a smaller carry-over loss for the LUL investor are realized. The consequence of this is a faster convergence between the FUL and LUL total equity-to-wealth ratios. Both the Value and Growth portfolio (Figure 9) and the Small and Large portfolio (Figure 10) show that the FUL and LUL total equity-to-wealth ratios start converging after 2009, which is largely driven by the diminishing of the carry-over loss to zero. On the contrary, the backtest using the S&P 500 Index over the same period shows no sign of convergence (Figure 3) because the accumulated carry-over loss is large enough to cover the entire backtest window (Figure 5).

Overall, the evolution of stock specific and total equity-to-wealth ratios, and for the other quantities discussed, of the Value and Growth two-stock portfolio from 1999 to 2014 is similar to the Small and Large backtest over the same period. What we can add is that significant increases in the basis-to-price ratio of one or both stocks can lead to significant rebalancing in the portfolio. By 2007, both the LUL and the FUL investor have a large tilt to Value in their portfolio. Selling of Value is prevented by the lock-in through the low basis-to-price ratios. Buying of Growth is prevented by the high total equity-to-wealth ratio. From the bottom left and bottom middle plots of Figure 10, we see that the LUL and FUL investors buy a tiny amount of Value stock but a large amount of Growth stock after the 2008 stock market decline. Without the stock market decline, which drives down the total equity-to-wealth ratio, such a large, diversification driven, rebalancing would involve substantial tax trading costs in Value or an overexposure to equity.

In the Internet Appendix E, we include cases with two stocks instead of two portfolios of stocks to study examples with higher volatility. These examples, using 0% correlation, show that simultaneous capital gains and losses of individual stocks do not reduce the effects of carry-over losses for an investor's portfolio choice.

3.10 Backtesting with Income

In the U.S. tax code, individual investors can offset up to $3,000 of taxable income per year with realized capital losses. Since incorporating this tax provision requires keeping track of wealth as an extra state variable, we assume that income grows deterministically at a constant rate of 3% per year, in line with the assumptions in Viceira (2001). We label this investor as LUL3K. A description of the partial equilibrium setting with income is given in the Internet Appendix D.

Figure 11 contains six plots that summarize our findings regarding the $3,000 capital gain tax provision through the average backtested path of equity-to-wealth ratios by investor age, as in the bottom left plot in Figure 3.[25] In the plots in the left column, income at age 20 is $50,000 ($Y_{20} = 50K$); in the plots in the right column, income at age 20 is $100,000 ($Y_{20} = 100K$). While income at age 20 can be much lower than $50,000, it is not very common that a single household, let alone a family, with income significantly below $50,000 participates actively in the stock market in a taxable account. The wealth-to-income ratios

[25] Plots of realized paths of equity-to-wealth ratios of selected backtest windows for individual investors are presented in the Internet Appendix.

(W_{20}/Y_{20}) at age 20 are 0, 2, 5, and 1,000 in the top, middle top, middle bottom, and bottom plots, respectively. We use the first three wealth-to-income levels for the plausible case when at age 20 an investor has not yet accumulated a significant amount of wealth. A wealth-to-income ratio of 1,000 may be achievable through inheritance or a similar form of intergenerational wealth transfer.

The \$3,000 capital gain tax provision allows the LUL3K investor to claim an income tax reduction of up to \$1,050, assuming a marginal income tax rate of 35%, or equivalently a capital gain tax rebate of up to \$1,050. Such a tax rebate encourages the LUL3K investor to hold more stocks. This effect is most significant for investors with low income and without financial wealth, as shown in the top left plot in Figure 11, where the 20-year-old investor starts out without wealth and \$50,000 income per year. The increase in the equity-to-wealth ratio is roughly one third of the difference between the FUL and LUL equity-to-wealth ratio. However, the difference between the LUL3K and LUL equity-to-wealth ratio declines rapidly over time as the investor starts to accumulate wealth. For investors who start with higher income, such as \$100,000 per year, or with higher initial wealth, such as twice, or even five times, the income at age 20, the effect of the \$3,000 tax provision on their equity-to-wealth ratios are less pronounced and also decrease faster over time. With a wealth-to-income ratio of 1,000, investor's income and the \$3,000 capital gain tax provision on income are both negligible as compared to the size of wealth. Hence, the investor's portfolio choice is very close to the case without income.

4 Conclusion

We integrate the limited use of losses assumption into multiple stock portfolio problems with capital gain taxation and show that requiring that capital losses can only be used to offset current or future realized gains significantly changes the after-tax risk-return tradeoff of holding equity. In contrast, a full use of losses investor's tax rebates act as an income process in down markets leading to a higher demand for equity relative to an untaxed investor when capital gains are not too large in the existing portfolio.

Without a carry-over loss, the FUL no-trade region is significantly above the LUL no-trade region over a large range of basis-to-price ratios close to one. With some carry-over losses, this range extends to moderate and low basis-to-price ratios as well. Through backtesting, using the S&P 500 Index, Value and Growth or Small and Large portfolios, we confirm that carry-over losses drive down the upper boundaries of the limited use of losses no-trade region. On average, we see that optimal equity holdings with limited use of losses are over an extended period of time significantly lower than the case with full use of losses.

We close by discussing what might be wrong with the intuition that simultaneous capital gains and losses in a multi-stock setting allow for portfolio rebalancing close to the unconstrained optimal portfolio without any embedded capital gains or losses. In all backtests, we see that the more there are simultaneous capital gains and losses in the portfolio, the closer the optimal equity allocation is to the conditional allocation with no embedded gains or losses, where the difference between the FUL and LUL equity-to-wealth ratios is the largest. Therefore, simultaneous capital gains and losses tend to amplify, not reduce, the effect of the limited use of losses. Since our evidence is only two stock (two portfolio) based, we save this question for future work.

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Figure 2: **Conditional Holdings.** The figure shows optimal portfolios conditional on the beginning period equity-to-wealth, basis-to-price ratios, and age implied by S&P 500 Index returns over the period of 1927 to 2013. Conditional holdings are expressed as an equity-to-wealth ratio, for an FUL and an LUL investor. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The top left plot shows hypothetical unconstrained optimum (red dashed line) and starting portfolio (red dot with black circle) with a basis-to-price ratio of 1, buy region, no-trade region, sell region, and wash-sell region as a function of the basis-to-price ratio. The top right plot shows the no-trade regions for an FUL (green dashed lines) and an LUL (blue solid lines) investor with zero carry-over loss when the investors are 20-year-old (dark) and 80-year-old (light), respectively. The bottom left plot shows the trading regions A, B, C, and D for an FUL (dark green dashed line) and an LUL (dark blue solid line) investor with zero carry-over loss at age 20. The bottom right plot shows the no-trade regions at age 20 for an FUL investor and an LUL investor with a carry-over loss of 0% of wealth (dark blue solid line), 1% of wealth (red dashed line), 10% of wealth (beige dotted line), and ∞ (INF) (light green solid line). Tax rates are set to match the U.S. tax code (see Subsection 3.1). Additional information regarding optimal portfolios conditional on the beginning period equity-to-wealth, basis-to-price ratios, and age implied by S&P 500 Index returns over the period of 1927 to 2013 are summarized in the Internet Appendix.

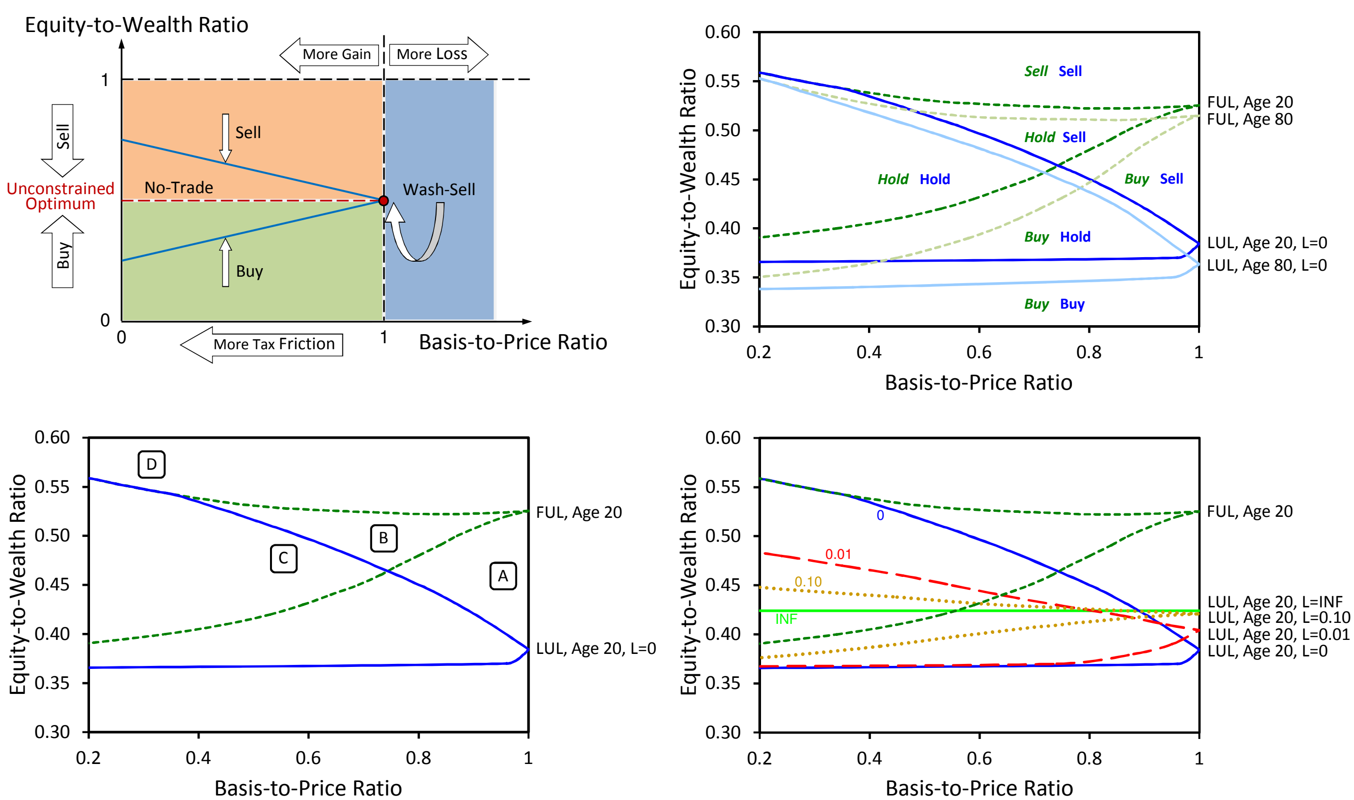

Figure 3: **Tax Trading Strategies using the S&P 500 Index Backtest.** The figure reports backtests of the optimal trading strategy, expressed as an equity-to-wealth ratio, for an FUL, an LUL, and an NCGT investor. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The investor's tax trading strategy along the historical path of the S&P 500 Index assumes that the backtests start at the beginning of the first year of each backtest window when the investor is 20-year-old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. The top left and middle left plots show allocations over 30 years with 31 optimal trading strategies starting in 1929 and 1965, respectively. The top right, middle right, and bottom right plots show allocations over 15 years with 16 optimal trading strategies starting in 1950, 1980, and 1999, respectively. The bottom left plot shows the equity-to-wealth ratio averaged over 58 investors who start investing in 1927, 1928, ..., and 1984, respectively, when they are 20-year-old. The 58th investor enters the equity market in 1984 at age 20 and reaches age 50 in 2014. Tax rates are set to match the U.S. tax code (see Subsection 3.1). Optimal portfolios conditional on the beginning period equity-to-wealth, basis-to-price ratios, and age implied by S&P 500 Index returns over the period of 1927 to 2013 are summarized in the Internet Appendix.

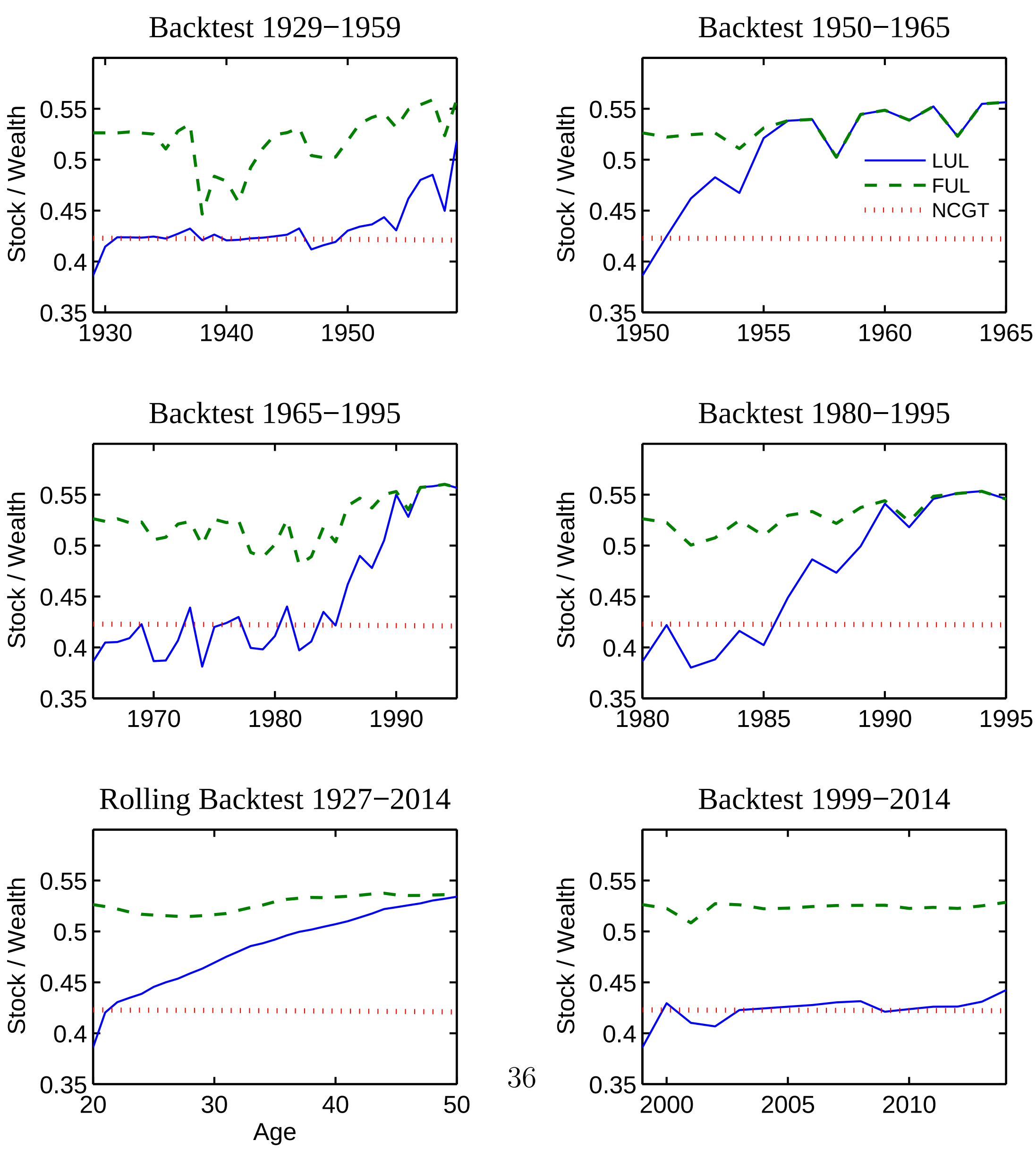

Figure 4: **No-trade Regions using the S&P 500 Index.** The figure reports entering and exiting equity-to-wealth ratios for an FUL and an LUL investor for the backtested periods starting in 1950 and 1999 (see Figure (3)) conditional on the basis-to-price ratio. The area inside the dark (light) solid blue line shows the no-trade region for the LUL investor at age 20 (80). The area inside the dark (light) green dashed line shows the no-trade region for the FUL investor at age 20 (80). The optimal strategy implies that above (below) the no-trade region investors sell (buy) equity shares to trade onto the boundary of the no-trade region. The right plots showing the backtest starting in 1999, present the no-trade region for the LUL investor at age 20 conditional on a carry-over-loss to wealth ratio of 0 (blue solid line), 0.01 (red dashed line), and 0.1 (beige dotted line), respectively.

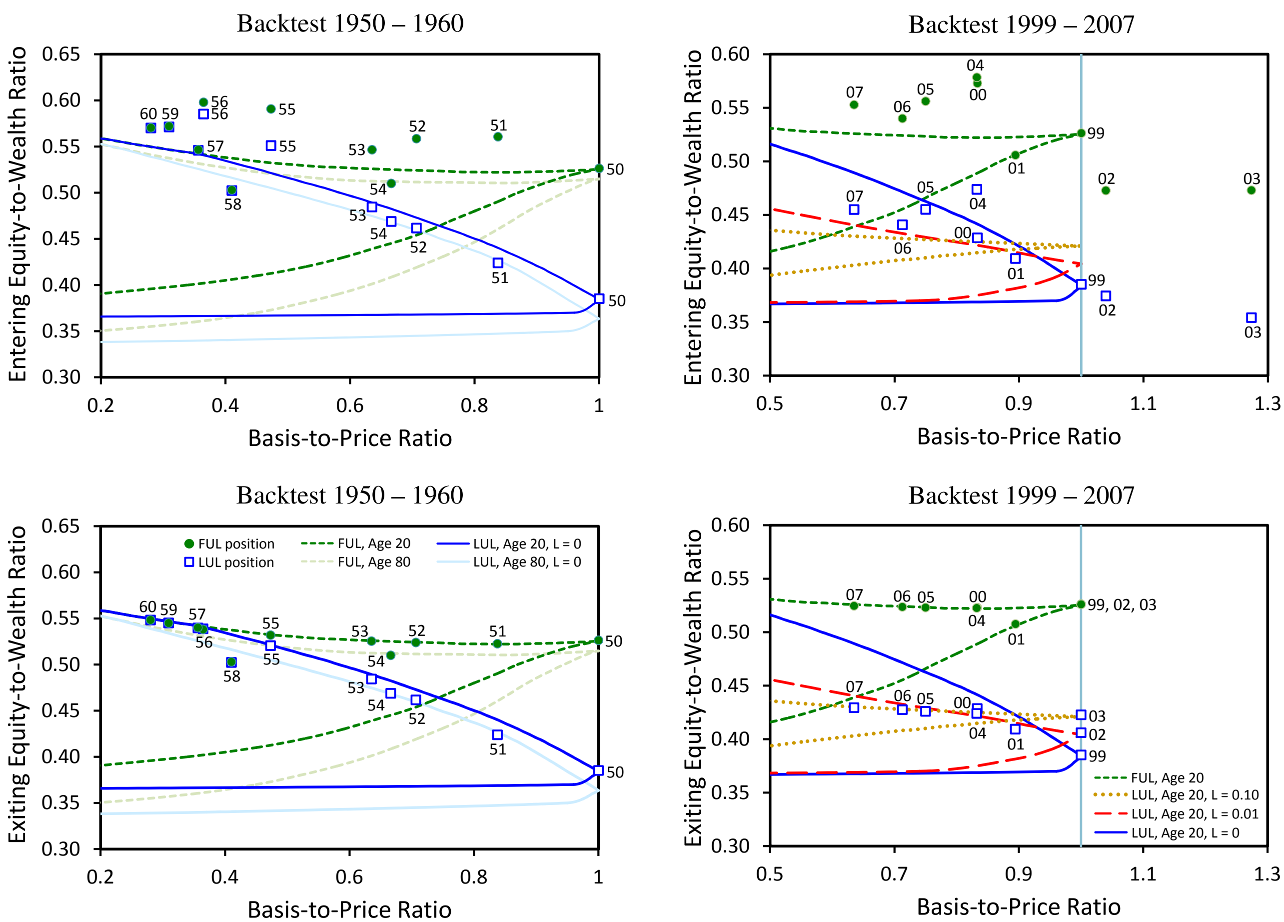

Figure 5: **Taxes and Carry-Over Losses using the S&P 500 Index.** The figure reports backtests of the tax basis, the carry-over-loss-to-wealth ratio, and cumulative capital gain taxes for an FUL, an FUL(LUL) (not shown when identical to FUL), an LUL, and an NCGT investor implied by the tax trading strategy for the periods starting in 1950 and 1999 (see Figure (3)). Figures corresponding to the other backtested periods shown in Figure (3) are delegated to the Internet Appendix.

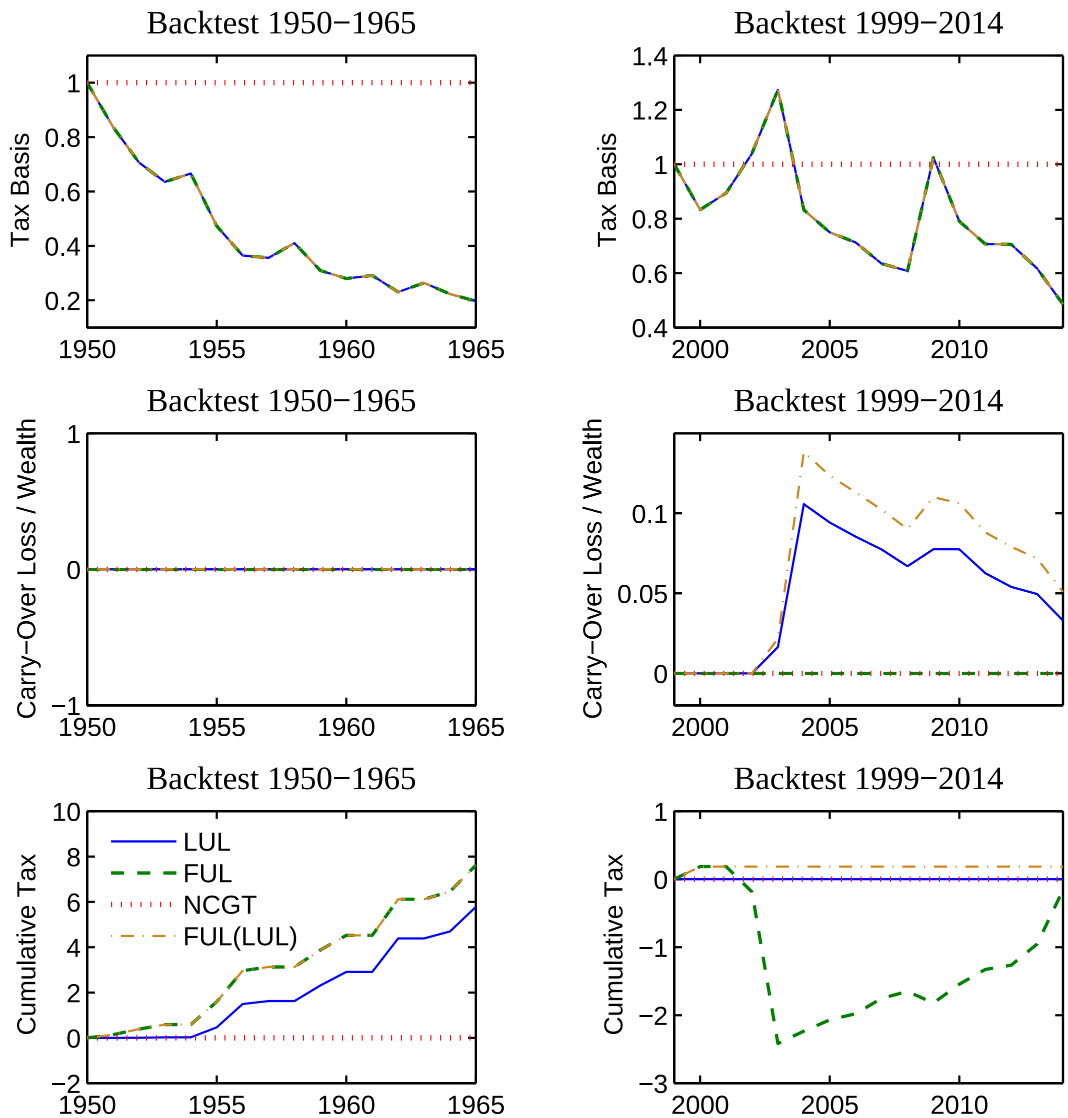

Figure 6: **Consumption and Wealth Implied by Tax Trading Strategies using the S&P 500 Index.** The figure reports backtests of wealth, consumption, wealth scaled by NCGT wealth, consumption scaled by NCGT consumption, wealth volatility and consumption volatility of an FUL, an FUL(LUL) (not shown when identical to FUL), an LUL, and an NCGT investor implied by the tax trading strategy for the period starting in 1999 (see Figure (3)). Volatility is computed by estimating a GARCH(1,1) model with a constant mean term. Figures corresponding to the other backtested periods shown in Figure (3) are presented in the Internet Appendix.

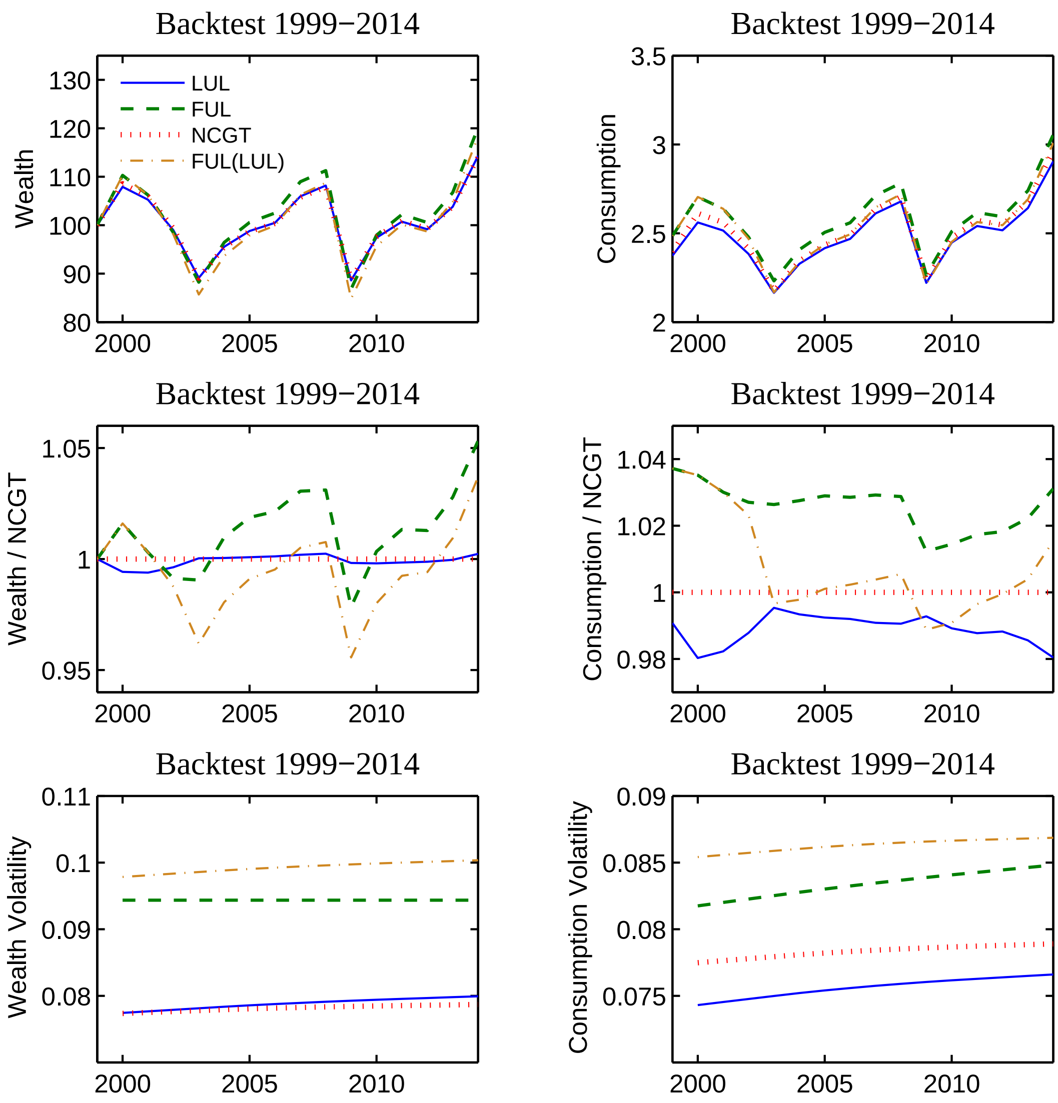

Figure 7: **The Costs of the Tax Treatment of Capital Losses.** The figure reports the average cumulative utility and the wealth equivalent relative to the NCGT investor for an FUL, an FUL(LUL), an LUL, and an NCGT investor. The average cumulative utility and the wealth equivalent are averaged over 50,000 simulation paths sampled from the S&P 500 Index. In the bottom plots, the average cumulative utility and the wealth equivalent are averaged over NBER dated recessions, where we jointly sample from the S&P 500 Index and NBER recessions. Each path has 80 periods (age 20 to 100). The Tax rates are set to match the U.S. tax code (see Subsection 3.1).

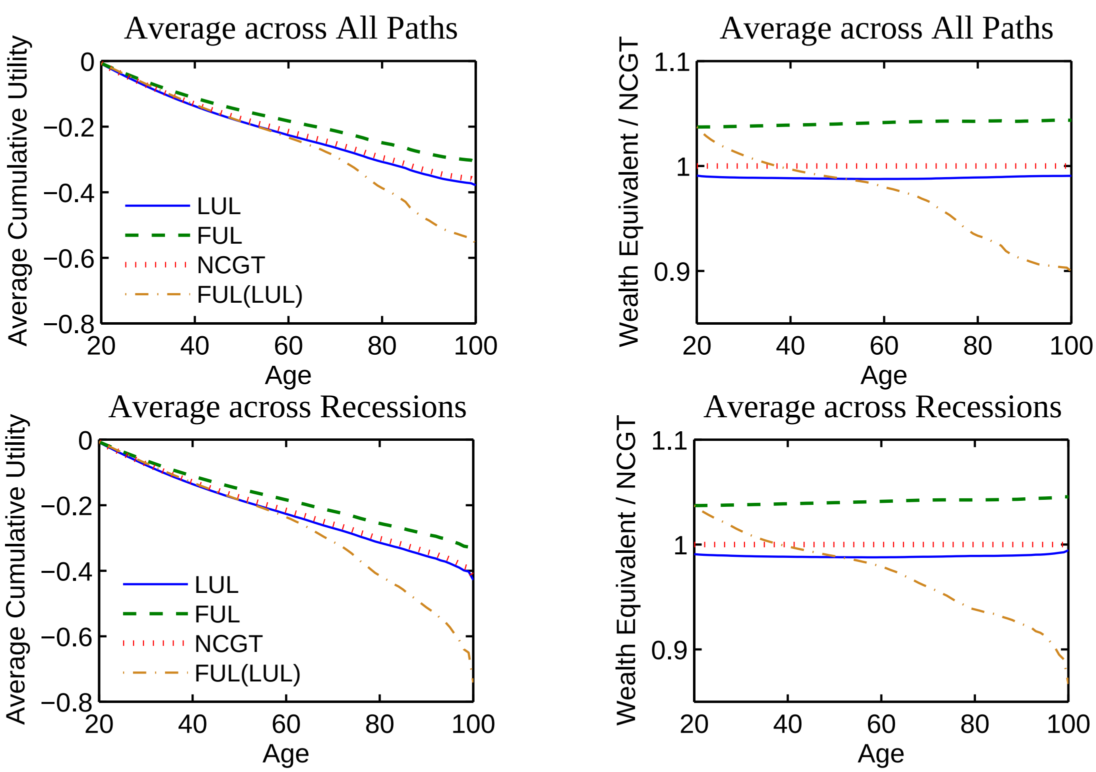

Figure 8: **Tax Trading Strategies using Small and Large I Backtest.** The figure reports backtests of the optimal trading strategies (Small-to-wealth and Large-to-wealth plus total equity-to-wealth ratios), optimal tax bases, carry-over-loss-to-wealth ratio, stock shares, and cumulative capital gain taxes for an FUL, an FUL(LUL) (not shown when identical to FUL), an LUL, and an NCGT investor. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The investor's tax trading strategy along the historical path of Small stocks and Large stocks assumes that the backtest starts at the beginning of 1965 when the investor is 20-year-old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. Additional backtested optimal Small and Large strategies are in the Internet Appendix. Tax rates are set to match the U.S. tax code (see Subsection 3.1).

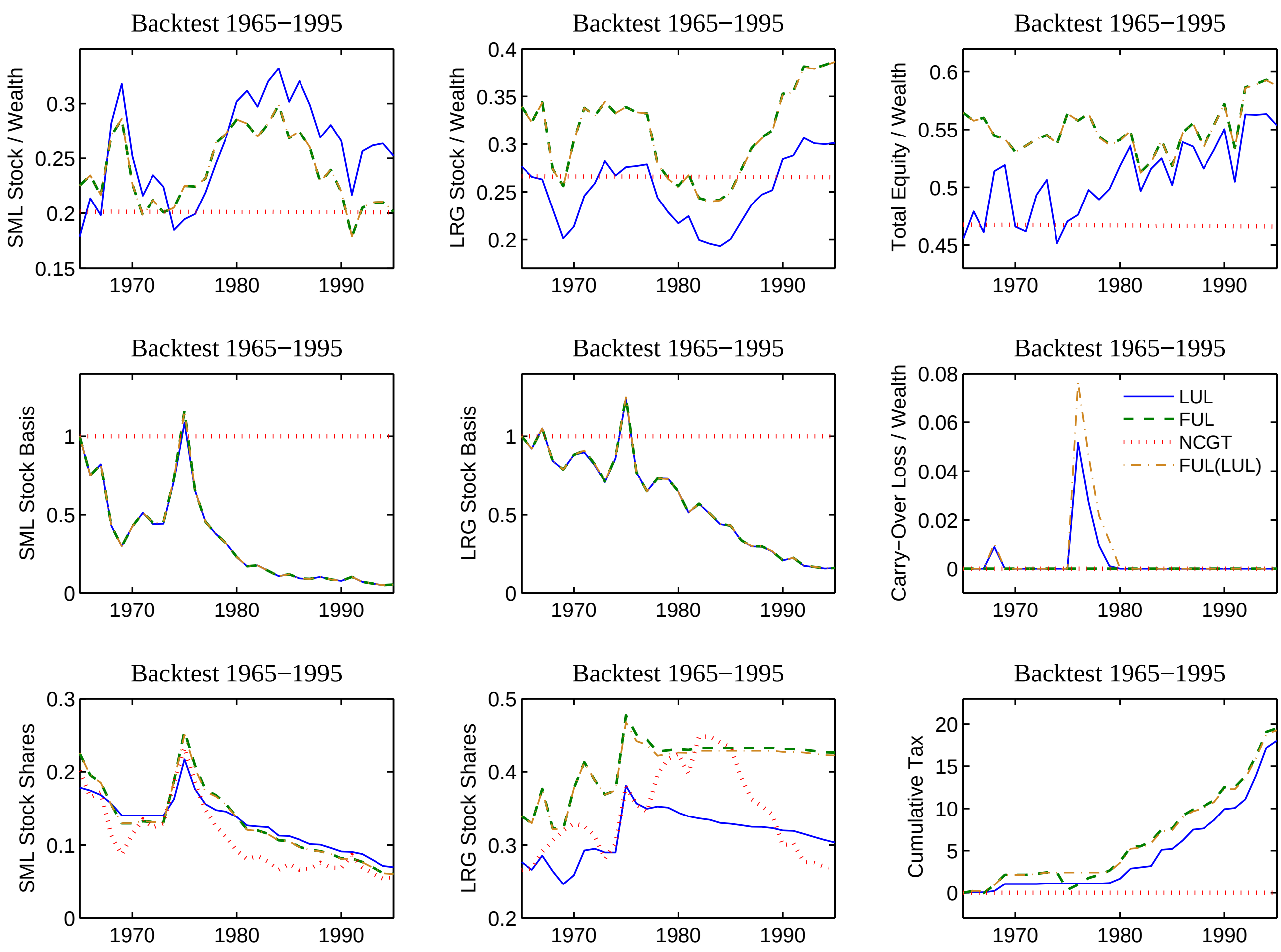

Figure 9: **Tax Trading Strategies using Small and Large II Backtest.** The figure reports backtests of the optimal trading strategies (Small-to-wealth and Large-to-wealth plus total equity-to-wealth ratios), optimal tax bases, carry-over-loss-to-wealth ratio, stock shares, and cumulative capital gain taxes for an FUL, an FUL(LUL) (not shown when identical to FUL), an LUL, and an NCGT investor. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The investor's tax trading strategy along the historical path of Small stocks and Large stocks assumes that the backtest starts at the beginning of 1999 when the investor is 20-year-old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. Additional backtested optimal Small and Large strategies are in the Internet Appendix. Tax rates are set to match the U.S. tax code (see Subsection 3.1).

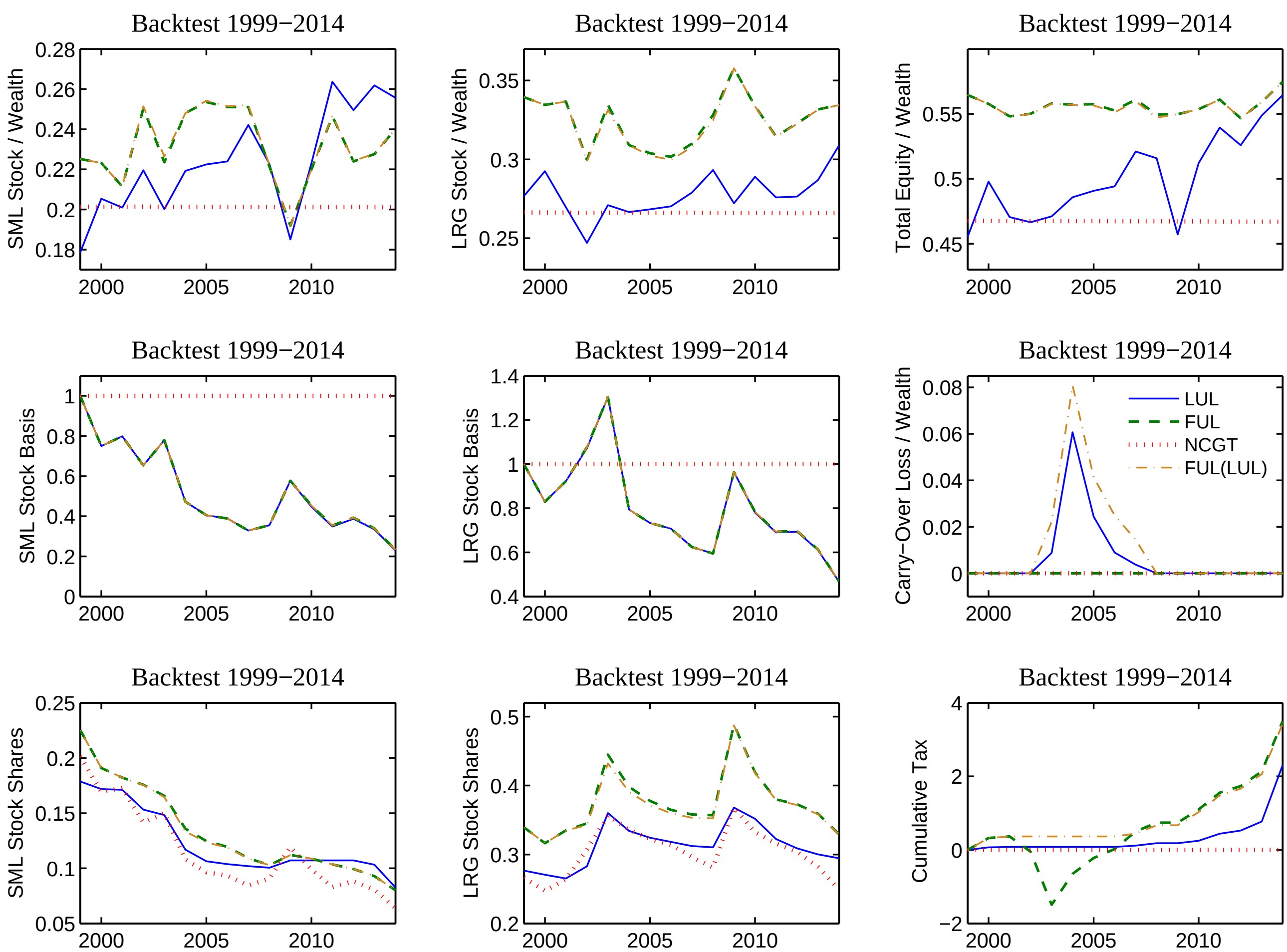

Figure 10: **Tax Trading Strategies using Value and Growth Backtest.** The figure reports backtests of the optimal trading strategies (Value-to-wealth and Growth-to-wealth plus total equity-to-wealth ratios), optimal tax bases, carry-over-loss-to-wealth ratio, stock shares, and cumulative capital gain taxes for an FUL, an FUL(LUL) (not shown when identical to FUL), an LUL, and an NCGT investor. Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The investor's tax trading strategy along the historical path of Value and Growth assumes that the backtest starts at the beginning of 1999 when the investor is 20-year-old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. Tax rates are set to match the U.S. tax code (see Subsection 3.1).

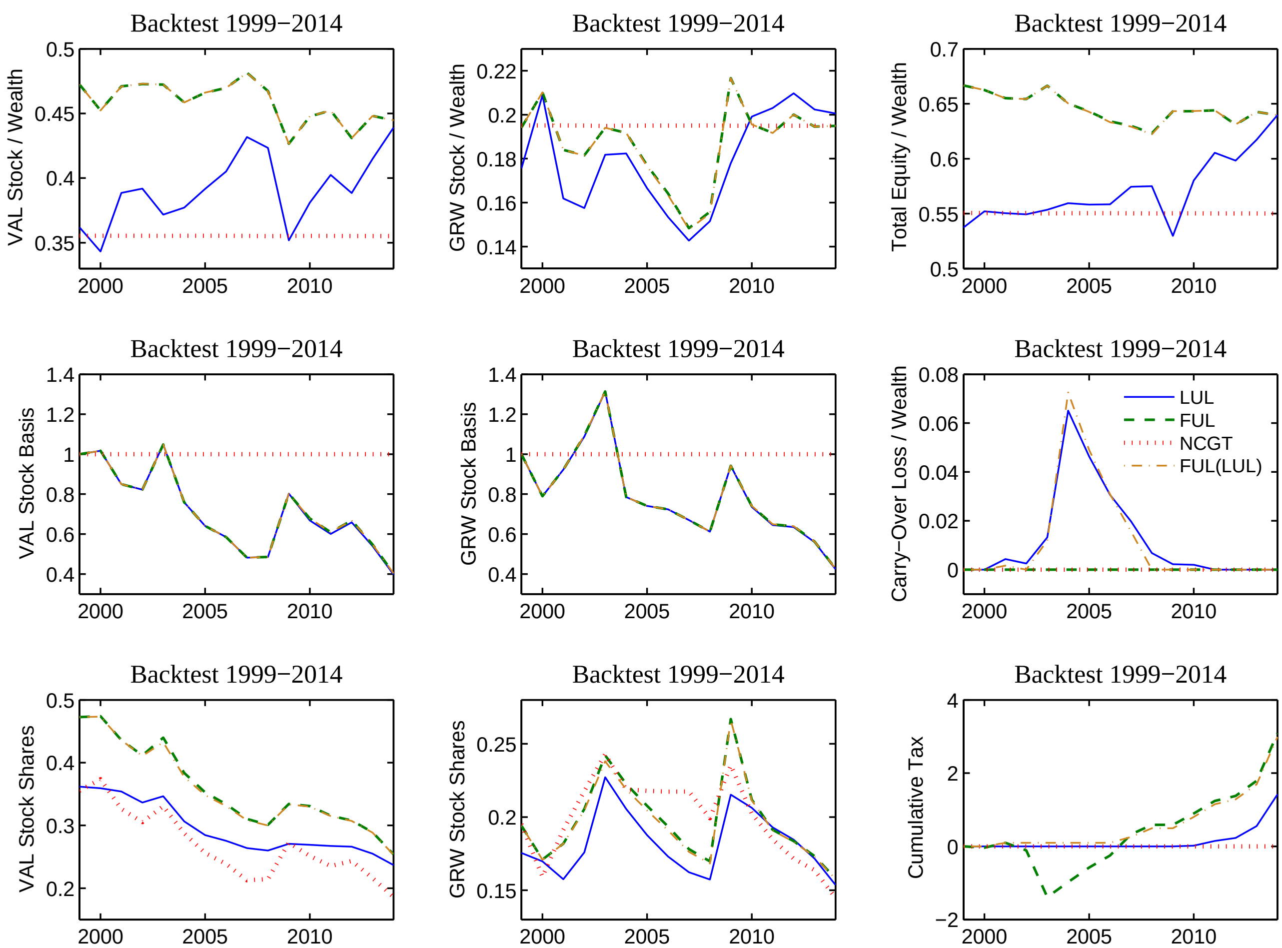

Figure 11: **Tax Trading Strategies with Income.** The figure reports averaged backtests of the optimal trading strategy with income, expressed as an equity-to-wealth ratio, for an FUL, an LUL, an NCGT investor, and an investor who can offset up to $\$3,000$ of taxable income per year with realized capital losses (LUL3K). Equity-to-wealth ratios are computed by dividing the value of the equity position by the value of the investor's wealth after subtracting dividend and interest taxes, but before consumption and capital gains taxes. The investor's tax trading strategy along the historical path of the S&P 500 Index assumes that the backtests start at the beginning of the first year of each backtest window when the investor is 20-year-old with initial stock position of 0, initial tax basis of 1, and initial carry-over loss of 0. The equity-to-wealth ratios are averaged over 58 investors who start investing in 1927, 1928, ..., and 1984, respectively, when they are 20-year-old. The 58th investor enters the equity market in 1984 at age 20 and reaches age 50 in 2014. Income, Y, and wealth, W, are in thousands of dollars. Income grows deterministically at a constant rate of 3% per year as in Viceira (2001). Tax rates are set to match the U.S. tax code (see Subsection 3.1). Optimal portfolios conditional on the beginning period equity-to-wealth, basis-to-price ratios, and age, implied by S&P 500 Index returns over the period of 1927 to 2013, are summarized in the Internet Appendix.